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CONTROL OF A FOURTH-ORDER LINEAR SYSTEM WITH MIXED CONSTRAINTS†

I. M. ANAN'YEVSKII

Moscow

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A controlled fourth-order linear mechanical system, containing a vibrating member, is considered. Geometric constraints are imposed on the control and phase variables. The problem of bringing the system to a given state in a finite time is solved. The solution employs an approach based on Kalman's general scheme for constructing controls as linear combinations of characteristic motions of the uncontrolled system. Results of a numerical simulation of the dynamics of a closed system are presented. © 2001 Elsevier Science Ltd. All rights reserved.

We consider the problem of damping the vibrations of a load attached at the end of an elastic beam, using an active dynamic damper with a translating mass. The control variable will be the interaction force between the damper and the load. Systems of this type are used, for example, in spacecraft (SC), , where measuring devices are mounted on a platform (P) at a considerable distance from the main body of the spacecraft, using a long rod. High accuracy in positioning and stabilizing the measuring instruments is required in order to perform measurements; hence; it is of paramount importance to damp any vibrations of the rod, and this must be taken into consideration in spacecraft design. One way to solve the problem is to use a controllable damper situated on the platform itself. The damper consists of guide 1 perpendicular to the axis of the rod 2, and a movable mass 3 which can be displaced along the guide by an electric drive (Fig. 1). This scheme is suitable for damping transverse vibrations of the rod.

A particular feature of this problem is the presence of two natural constraints on the different variables of the system. One, due to the restricted possibilities of the drive, is imposed on the control force; the other, due to the finiteness of the path of the damper mass (the damper guide is finite in size), is imposed on the displacement of the second mass relative to the first.

Thus, the problem may be classed as a problem with mixed constraints, that is, constraints imposed on different variables of the system. To solve it, we propose to use an approach based on Kalman's general scheme for constructing controls as linear combinations of characteristic motions of the uncontrolled system [2]. This approach has been extended to the case where the control function is subject to constraints [3, 4].

1. FORMULATION OF THE PROBLEM

Under certain simplifying assumptions [1], the following two-mass mechanical control system containing a vibrating member (Fig. 2) may serve as a model for the structures just described. Two bodies, of masses m_1 and m_2, move along a horizontal straight line. The first body is connected to a fixed base by a spring of stiffness $k > 0$. The second body is connected to the first by a drive which generates a force u. The equations of motion of the system are

$$m_1\ddot{y} + ky = -u, \quad m_2\ddot{z} = u \tag{1.1}$$

where y is the coordinate of the first body and z is the coordinate of the second body on the straight line. The following constraint is imposed on the control force u

$$|u(t)| \leq a, \quad a > 0 \tag{1.2}$$

and the displacement of the first body relative to the first must satisfy the condition

$$|z(t) - y(t)| \leq d, \quad d > 0 \tag{1.3}$$

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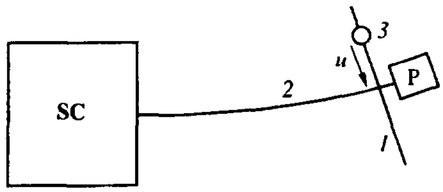

Fig. 1

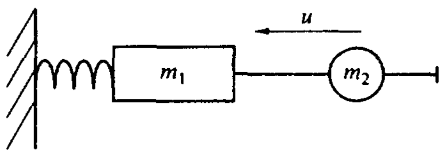

Fig. 2

It is required to construct a control $u(t)$ which satisfies constraint (1.2) and which brings system (1.1) from the given initial state

$$y(0) = y^0, \quad \dot{y}(0) = \dot{y}^0, \quad z(0) = z^0, \quad \dot{z}(0) = \dot{z}^0 \tag{1.4}$$

to the rest state

$$y(T) = z(T) = 0, \quad \dot{y}(T) = \dot{z}(T) = 0 \tag{1.5}$$

The coordinates $y(t)$ and $z(t)$ must satisfy condition (1.3) throughout the whole motion, whose completion time T is not fixed.

We introduce the new variables

$$x_1 = ky, \quad x_3 = m_2kz/m_1, \quad t' = (k/m_1)^{1/2}t \tag{1.6}$$

In terms of the new variables, system (1.1) becomes

$$\ddot{x}_1 + x_1 = -u, \quad \ddot{x}_3 = u \tag{1.7}$$

and constraint (1.3) has the form $|m_1x_3/m_2 - x_1| \leqslant kd$. We introduce a constant vector $p^+ = (-1, 0, m_1/m_2, 0)$ and rewrite the last inequality as follows:

$$|p^\tau x(t)| \leqslant kd \tag{1.8}$$

The dots in Eqs (1.7) and later denote derivatives with respect to the new time t'. Henceforth the prime on t' will be omitted.

Put $\dot{x}_1 = x_2$, $\dot{x}_3 = x_4$. After the change of variables (1.6), conditions (1.4) and (1.5) become

$$x_i(0) = x_i^0, \quad x_i(T) = 0, \quad i = \overline{1, 4} \tag{1.9}$$

where x_i^0 are certain given constants and $T > 0$ is the as yet unknown completion time of the process.

The problem reduces to constructing a control that will satisfy constraint (1.2) and bring system (1.7) from the given initial state (1.9) to the origin while observing constraint (1.8) throughout the whole motion.

2. CONSTRUCTION OF THE CONTROL

Let $x = (x_1, x_2, x_3 = x_4)$ denote the phase vector of system (1.7); we rewrite the system in vector form

$$\dot{x} = Ax + bu \tag{2.1}$$

$$A = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{Vmatrix}, \quad b = \begin{Vmatrix} 0 \\ -1 \\ 0 \\ 1 \end{Vmatrix}$$

The initial and final conditions (1.9) may be written as follows:

$$x(0) = x^0, \quad x(T) = 0 \tag{2.2}$$

Following Kalman's approach [2], we will seek the control as a linear combination of characteristic motions of the homogeneous (uncontrolled) system (2.1)

We introduce the notation

$$s = \sin t, \quad c = \cos t \tag{2.3}$$

The fundamental matrix of the solutions of the homogeneous system and its inverse have the form

$$\Phi(t) = \begin{Vmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{Vmatrix}, \quad \Phi^{-1}(t) = \begin{Vmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

Consider the vector

$$h(t) = \Phi^{-1}(t)b, \quad h^{\mathrm{T}}(t) = (s, -c, -t, 1) \tag{2.4}$$

and the matrices

$$Q(t) = h(t)h^{\mathrm{T}}(t) = \begin{Vmatrix} s^2 & -sc & -ts & s \\ -sc & c^2 & tc & -c \\ -ts & tc & t^2 & -t \\ s & -c & -t & 1 \end{Vmatrix}$$

$$R(t) = \int_0^t Q(\tau)d\tau = \begin{Vmatrix} (t-sc)/2 & -s^2/2 & tc-s & 1-c \\ -s^2/2 & (t+sc)/2 & ts+c-1 & -s \\ tc-s & ts+c-1 & t^3/3 & -t^2/2 \\ 1-c & -s & -t^2 2 & t \end{Vmatrix} \tag{2.5}$$

The expression for the control function $u(t)$ which brings system (2.1) from the initial state x^0 to the origin of the phase space may be written as follows [2]:

$$u(t) = V^{\mathrm{T}}(t, T)x^0, \quad V(t, T) = -R^{-1}(T)h(t) \tag{2.6}$$

3. SATISFACTION OF THE CONSTRAINTS ON THE CONTROL FUNCTION

We will show that, if the completion time T of the process is taken to be fairly long, one can guarantee that the control $u(t)$ will satisfy constraints (1.2). To that end, we estimate the function $u(t)$ as follows:

$$|u(t)| \leq \sum_{i=1}^{4} |V_i(t, T)x_i^0| \leq \|V(t, T)\|_\infty \|x^0\|_1 \tag{3.1}$$

where $\|\cdot\|_\infty$ and $\|\cdot\|_1$ denote the norms in the spaces R_∞^4 and R_1^4, respectively, which have the following form for an arbitrary vector q

$$\| q \|_\infty = \max_{1 \leqslant i \leqslant 4} | q_i |, \quad \| q \|_1 = \sum_{i=1}^{4} | q_i |$$

We introduce the auxiliary function

$$v(T) = \max_{0 \leqslant t \leqslant T} \| V(t, T) \|_\infty \tag{3.2}$$

and rewrite estimate (3.1) as follows:

$$\max_{0 \leqslant t \leqslant T} | u(t) | \leqslant v(T) \| x^0 \|_1 \tag{3.3}$$

We propose two ways of determining the completion time of the motion for which constraints (1.2) will be observed. The first is based on an analytical estimate of the function $v(T)$ and the second on a numerical construction of the function.

We first choose T to be $T = 2\pi n$, where n is a natural number. In that case the matrix $R(T)$ is simplified:

$$R(T) = \begin{Vmatrix} T/2 & 0 & T & 0 \\ 0 & T/2 & 0 & 0 \\ T & 0 & T^3/3 & -T^2/2 \\ 0 & 0 & -T^2/2 & T \end{Vmatrix} \tag{3.4}$$

and its inverse is

$$R^{-1}(T) = \frac{1}{\Delta} \begin{Vmatrix} 2T & 0 & -24/T & -12 \\ 0 & 2\Delta/T & 0 & 0 \\ -24/T & 0 & 12/T & 6 \\ -12 & 0 & 6 & 4(T^2-6)/T \end{Vmatrix}, \quad \Delta = T^2 - 24 \tag{3.5}$$

Let us write down the components of the vector function $V(t, T)$, using the last expression for the matrix $R^{-1}(T)$ and formula (2.4); allowing for the fact that $T \geqslant 2\pi$, we can estimate them as follows:

$$| V_1(t, T) | = \frac{| 2T^2 \sin t + 24t - 12T |}{T\Delta} \leqslant \frac{2T + 12}{\Delta} \leqslant \frac{4T}{\Delta}$$

$$| V_2(t, T) | = \frac{| 2\cos t |}{T} \leqslant \frac{2}{T} \leqslant \frac{4T}{\Delta}$$

$$| V_3(t, T) | = \frac{| -24 \sin t - 12t + 6T |}{T\Delta} \leqslant \frac{6T + 24}{T\Delta} \leqslant \frac{4T}{\Delta}$$

$$| V_4(t, T) | = \frac{| -12 \sin t - 6Tt + 4T^2 - 24 |}{T\Delta} \leqslant \frac{4T^2 - 12}{T\Delta} \leqslant \frac{4T}{\Delta}$$

These estimates and definition (3.2) of the function $v(T)$ imply that $v(T) \leqslant 4T/\Delta$. Hence, using (3.3), we obtain an estimate for the control function $u(t)$

$$\max_{0 \leqslant t \leqslant T} | u(t) | \leqslant \frac{4T}{T^2 - 24} \| x^0 \|_1 \tag{3.6}$$

Since $T = 2\pi n$, $n \in N$, it follows that for sufficiently large n

$$4T/(T^2 - 24) \leqslant a / \| x^0 \|_1 \tag{3.7}$$

Inequalities (3.6) and (3.7) guarantee that constraint (1.2) is satisfied.

We will describe another way of choosing the completion time T so as to ensure that the control function (2.6) will satisfy constraint (1.2). To that end, we construct the function $v(T)$ by numerical means, using relations (2.5), (2.6) and (3.2). The function $v(T)$ is uniquely defined by the matrix A and

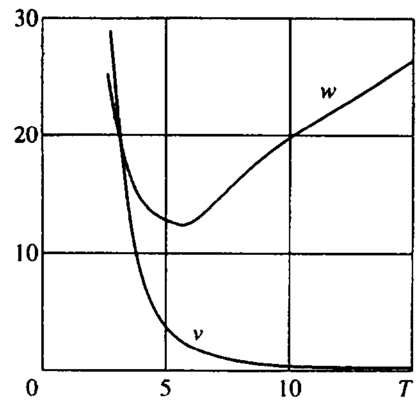

Fig. 3

the vector b of system (2.1), and therefore the construction may be carried out in advance once and for all for the given system.

Figure 3 shows a graph of the function $v(T)$. As might have been expected, it turned out that the maximum value of the control function was lower, the longer the time of motion of the system to its terminal state. As the completion time of the process one can choose any time T for which

$$v(T) \leq a/\|x^0\|_1 \tag{3.8}$$

4. SATISFACTION OF THE CONSTRAINTS ON THE PHASE COORDINATES

We now turn to the problem of choosing T so as to ensure that constraints (1.8) are satisfied. In the notation adopted above, the solution of system (2.1), beginning at time $t = 0$ from the point x^0, is

$$x(t) = \Phi(t)\left(x^0 + \int_0^t h(\tau)u(\tau)d\tau\right)$$

Substituting into this formula expression (2.6) for the control function $u(t)$ and using relations (2.5) defining the matrix $R(t)$, we obtain

$$\begin{aligned} x(t) &= \Phi(t)\left(x^0 - \int_0^t h(\tau)[h^{\top}(\tau)R^{-1}(T)x^0]d\tau\right) = \Phi(t)\left(x^0 - \left[\int_0^t h(\tau)h^{\top}(\tau)d\tau\right]R^{-1}(T)x^0\right) = \\ &= \Phi(t)(x^0 - R(t)R^{-1}(T)x^0) = W(t, T)x^0 \\ W(t, T) &= \Phi(t)[R(T) - R(t)]R^{-1}(T) \end{aligned} \tag{4.1}$$

As in the case of constraint (1.2) , we propose two ways of determining the values of T that will guarantee that condition (1.8) is satisfied.

We first choose the completion time of the process to be $T = 2\pi n$, $n \in N$. We estimate the Euclidean norm of the vector x in terms of the norms of the matrices on the right-hand side of (4.1) (by the norm of a matrix we mean the norm of the corresponding operator in Euclidean space).

It is well known [5] that $\|\Phi(t)\|^2$ is equal to the maximum eigenvalue $\varphi(t)$ of the matrix

$$\Phi^{\top}(t)\Phi(t) = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t^2+1 & t \\ 0 & 0 & t & 1 \end{Vmatrix}, \quad t \in [0, T]$$

It is not difficult to calculate that $\varphi(t) \leq t^2 + 2$, whence we obtain

$$\|\Phi(t)\| \leq (T^2+2)^{1/2} \tag{4.2}$$

The matrix $R^{-1}(T)$ is symmetric and positive-definite; consequently, its eigenvalues are positive and the maximum eigenvalue equals equals the norm of the matrix. In addition [5], the sum of all the eigenvalues equals the trace of $R^{-1}(T)$. The form (3.5) of the matrix $R^{-1}(T)$ implies that the number $2/T$ is one of its eigenvalues and

$$\frac{\operatorname{tr} R^{-1}(T)}{4} = \frac{2T^2-15}{T(T^2-24)} > \frac{2}{T}$$

Consequently, $2/T$ is not the maximum eigenvalue and

$$\|R^{-1}(T)\| \leq \operatorname{tr} R^{-1}(T) - \frac{2}{T} = \frac{6(T^2-2)}{T(T^2-24)}, \quad T = 2\pi n \tag{4.3}$$

By the definition of the matrix R

$$R(T) - R(t) = \int_t^T Q(\tau)d\tau = \int_t^T h(\tau)h^{\top}(\tau)d\tau$$

We will consider the functions $\sin\tau$, $-\cos\tau$, $-\tau$ and 1, the components of the vector function $h(\tau)$, as elements of the pre-Hilbert space $C_2[t, T]$ of functions, continuous in the intervals $[t, T]$, with scalar product

$$(f, g) = \int_t^T f(\tau)g(\tau)d\tau$$

Since the functions listed above are linearly independent, it follows that the matrix $(R(T)-R(t))$, which is their Gram matrix, is symmetric and positive-definite [6].

Similar arguments show that the matrices $R(t)$ and $R(T)$ are also symmetric and positive-definite, since they are Gram matrices of the same linearly independent functions considered as elements of the spaces $C_2[0, t]$ and $C_2[0, T]$, respectively. In addition

$$R(T) - R(t) < R(T), \quad 0 < t \leq T \tag{4.4}$$

(an inequality $X < Y$ for symmetric matrices X and Y means that the matrix $Y-X$ is positive-definite). Since the norm of a symmetric positive-definite matrix equals its maximum eigenvalue and inequality (4.4) implies the corresponding inequality for the eigenvalues of the matrix $R(T)-R(t)$ and $R(T)$[5], it follows that $\|R(T)-R(t)\| \leq \|R(T)\|$.

To estimate $\|R(T)\|$, we employ reasoning similar to that used above for the matrix $R^{-1}(T)$. It follows from the form (3.4) of the matrix $R(T)$ that the number $T/2$ is a non-maximum eigenvalue of $R(T)$, and therefore

$$\|R(T)\| \leq \operatorname{tr} R(T) - T/2 = T(2T^2+9)/6, \quad T = 2\pi n$$

It follows from (4.1)–(4.3), the equality $\|p\| = (m_1^2 + m_2^2)^{1/2}/m_2$ and the last relation that

$$|p^{\top}x(t)| \leq \|p\|\|\Phi(t)\|\|R(T)-R(t)\|\|R^{-1}(T)\|\|x^0\| \leq$$
$$\leq (m_1^2+m_2^2)^{1/2}(2T^2+9)(T^2-2)(T^2+2)^{1/2}\|x^0\|/(m_2(T^2-24))$$

Consequently, we can choose the completion time of the process to be those values of $T = 2\pi n$ for which

$$\frac{(2T^2+9)(T^2-2)(T^2+2)^{1/2}}{T^2-24} \leq \frac{m_2 kd}{(m_1^2+m_2^2)^{1/2}\|x^0\|} \tag{4.5}$$

There is another way to determine the value of T for which constraints (1.8) will be satisfied. It follows from (4.1) that

$$|p^{\top}x(t)| \leq |p^{\top}W(t,T)x^0| \leq w(T)\|x^0\|, \quad w(T) = \max_{0\leq t\leq T}\|W^{\top}(t,T)p\|$$

We will construct the function $w(T)$ numerically. As this function is uniquely defined by the matrix A and the vectors b and p, it suffices to carry out the construction just once.

Figure 3 shows graph of the function $w(T)$. Obviously, as the duration T of the process increases, so does $w(T)$, but not monotonically. As completion time of the motion one can take any T for which the value of $w(T)$ satisfies the inequality

$$w(T) \leq kd/\|x^0\| \tag{4.6}$$

5. COMPUTATION OF THE CONTROL AND NUMERICAL SIMULATION

Thus, the following procedure is proposed to construct the control function $u(t)$. First, given the initial state vector x^0, choose a value of T – the completion time of the process. The value of T may be sought in the form $T = 2\pi n$, where the natural number n must be such that conditions (3.7) and (4.5) are satisfied. Another way to determine T is through numerical construction of the functions $v(T)$ and $w(T)$ and the choice of the value of T for which inequalities (3.8) and (4.6) are satisfied.

After the completion time T of the process has been determined, the control function $u(t)$ is computed analytically using (2.6). The expression for the inverse matrix $R^{-1}(T)$, obtained by using the computer program MAPLE, turns out to be quite cumbersome. As an example, we present the element in the upper left corner of the matrix, using the notation (2.3)

$$\begin{aligned} r_{11} &= 2[(T^4 - 36T^2 + 72)sc + T(8T^2 - 9)c^2 + 36(T^2 - 2)s - \\ &-12T(T^2 - 6)c + T^5 - 14T^3 + 18][16T(T^2 + 15)sc + (T^4 - 96T^2 + 192)c^2 + \\ &+40T^2(T - 60)s - 8(T^4 - 6T^2 + 48)c + T^6 - 17T^4 + 48T^2 + 192]^{-1} \end{aligned}$$

Figure 4 presents the results of a numerical simulation of the dynamics of system (2.1). The system was steered from an initial state $x^0 = (0{,}5;\ -0{,}5;\ 0{,}5;\ 0{,}5)$ to the origin. The time required to complete the process was taken as $T = 10$. The solid curve is the projection of the phase trajectory on the plane $x_1, x_2 = \dot{x}_1$, while the thin curve is its projection on the plane $x_3, x_4 = \dot{x}_3$.

The solid curves in Fig. 5 are graphs of the control function $u(t)$ and the quantity

$$|p^{\top}x(t)| = |m_1x_3/m_2 - x_1| = k|z(t) - y(t)|$$

occurring in constraint (1.8) as functions of time, for the case $m_1/m_2 = 10$. For comparison, the dashed curves represent the same functions for a completion time $T = 5$. It is obvious that the quantity $|p^{\top}x(t)|$ is then reduced, while the maximum modulus of the control function $u(t)$ has increased.

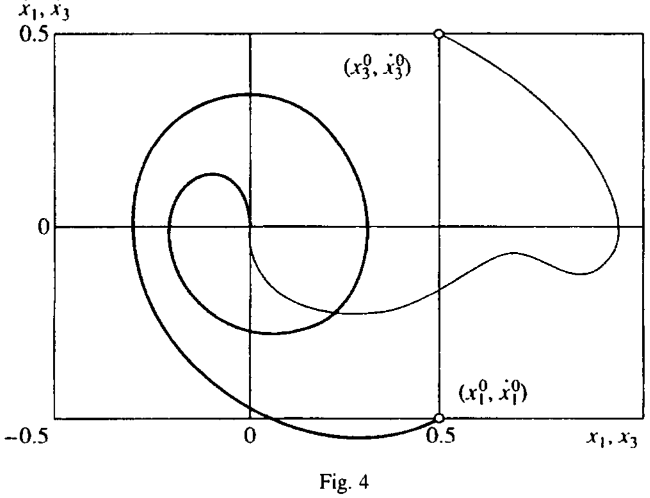

Fig. 4

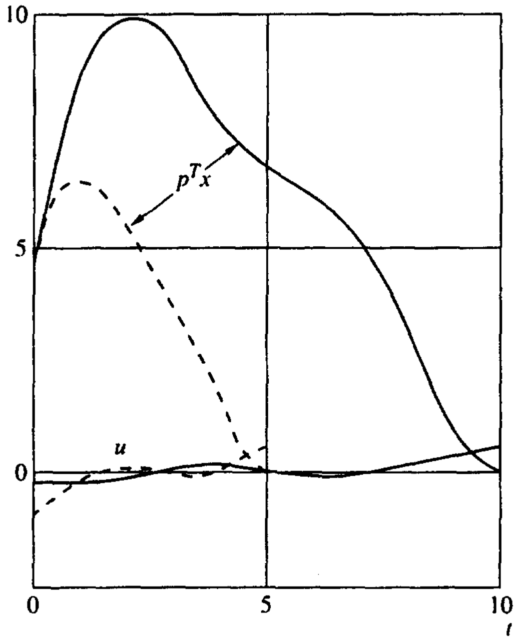

Fig. 5

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